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Amplitude equation for SPDE with quadratic nonlinearities

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Abstract

In this paper we rigorously derive stochastic amplitude equation for a rather general class of SPDEs with quadratic nonlinearities forced by small additive noise. Near a change of stability we show that the solution of the original SPDE is approximated by the solution of the amplitude equation. Our results significantly improve older results

We focus on equations with quadratic nonlinearity and give applications to the one-dimensional Burgers equation and a model from surface growth.

1 Introduction

Stochastic partial differentail equations (SPDE) with quadratic nonlinearities arise in various applications in physics. One example is the stochastic Burgers equation in the study of closure models for hydrodynamic turbulence [6]. Other examples are the growth of rough amorphous surfaces in the study of the growth of amorphous surfaces [24, 19], and the Kuramoto-Sivashinsky model, which originally models a fire front, but it is also used for surface erosion [7, 17]. All these models fit in the abstract framework of this paper.

Consider the following SPDE in Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$:

$$du = \left[Au + \varepsilon^2 Lu + B(u,u)\right] dt + \varepsilon^2 dW. \tag{1}$$

We consider (1) near a change of stability, where $\varepsilon^2\nu$ measures the distance from bifurcation. The operator A is assumed to be non-positive, and we call the kernel of A the dominant modes. We allow for noise given by a fairly general Q-Wiener process.

Near the bifurcation the equation exhibit two widely separated characteristic time-scales and it is desirable to obtain a simplified equation which governs the evolution of the dominant modes. This is well known on a formal level in many examples in physics (see e.g. [8]). Moreover, for deterministic PDEs on unbounded domains, this method [16, 20, 23, 12] successfully overcomes the gap of a lacking centre manifold theory. This is also usefull for SPDEs on bounded domains [3], where also no centre manifold theory is available.

Moreover, there are numerous variants of this method. However, most of these results are non-rigorous approximations using this type of formal multi-scale analysis. A noteable example is [9].

Another interesting question, which can be tackled with similar methods, is the stabilization effect due to degenerate noise. Here noise is transported via nonlinear interaction to the dominant modes. Examples are [21, 4, 13, 14, 15, 22]

The purpose of this paper is to derive rigorously an amplitude equation for a quite general class of SPDEs (cf. (1)) with quadratic nonlinearities. This work is based on [4], where degenerate noise in a different scaling was considered, and it improves significantly previously know results of [1], where in a similar situation much more regular noise was considered. A related result can be found in [2], where a simple multiplicative noise was considered, but again with much weaker results.

In this paper we focus on quadratic nonlinearities only. The case of cubic equations is much simpler, as one can rely on nonlinear stability. This case was already considered in [5], for instance.

As an application of our approximation result of Theorem 13, we discuss the stochastic Burgers equation and surface growth model. To illustrate our results consider the Burgers equation

$$\partial_t u = \left(\partial_x^2 + 1\right) u + \varepsilon^2 \nu u + u \partial_x u + \varepsilon^2 \partial_t W, \tag{2}$$

on $[0, \pi]$ subject to Dirichlet boundary conditions.

We show in our main result that near a change of stability on a time-scale of order ε^{-2} the solution of (2) is of the type

$$u(t,x) = \varepsilon b(\varepsilon^2 t) \sin(x) + \mathcal{O}(\varepsilon^2),$$

where b is the solution of the amplitude equation on the slow time-scale

$$\partial_T b(T) = \nu b(T) - \frac{1}{12} b^3(T) + \partial_T \beta(T),$$

with $\beta(T) = \varepsilon \alpha_1 \beta_1(\varepsilon^{-2} T)$ being a rescaled noise in $\mathcal{N}$.

This approximating equation is called amplitude equation, as it can by rewritten to an SDE for the amplitudes of an expansion of a with respect to a basis in $\mathcal{N}$.

For the proofs we rely on a cut-off technique, as in general we cannot control moments of solution and exclude the possibility of a blow up. Therefore all estimates are established only with high probability and not in moments. To be more precise, we use a stopping time, in order to look only at solutions that are not too large. Then we can use moments for time uniformly up to the stopping time. Later we use the amplitude equation itself to verify that the stopping is not small.

As the general strategy we first show that all non-dominant modes are given by an Ornstein-Uhlenbeck process and a quadratic term in the dominant modes. Then we rely on Itô -Formula and some averaging argument, in order to transform the equation for the dominant modes to an amplitude equation with an additional small remainder.

The rest of this paper is organised as follows. In Section 2 we state the assumptions that we make. In Section 3 we give a formal derivation of the amplitude equation and state the main results. In Section 4 we give the main results. Finally, in Section 5 we apply our theory to the stochastic Burgers equation and surface growth model.

2 Main Assumptions and Definitions

This section summarises all assumptions necessary for our results. For the linear operator A in (1) we assume the following:

Assumption 1 (Linear Operator A**)** *Suppose A is a non-positive operator on $\mathcal{H}$ with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_k \leq ....$ and $\lambda_k \geq Ck^m$ for all large k. the corresponding complete orthonormal system of eigenvectors is $\{e_k\}_{k=1}^{\infty}$ with $Ae_k = -\lambda_k e_k$.*

We use the notation $\mathcal{N} := \ker A$, $\mathcal{S} = \mathcal{N}^{\perp}$ the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}$, and P_c for the projection $P_c : \mathcal{H} \to \mathcal{N}$. Define, $P_s := I - P_c$, and suppose that P_c and P_s commute with A. Suppose that $\mathcal{N}$ has finite dimension n with basis $(e_1, ..., e_n)$.

As the dimension of $\mathcal{N}$ is finite, it is well known that both P_c and P_s are bounded linear operators on $\mathcal{H}$ (cf. Weidmann [25]).

Definition 2 *For $\alpha \in \mathbb{R}$, we define the space $\mathcal{H}^{\alpha}$ as*

$$\mathcal{H}^{\alpha} = \left\{ \sum_{k=1}^{\infty} \gamma_k e_k : \sum_{k=1}^{\infty} \gamma_k^2 \ k^{2\alpha} < \infty \right\} \quad \textit{with norm} \ \left\| \sum_{k=1}^{\infty} \gamma_k e_k \right\|_{\alpha}^2 = \sum_{k=1}^{\infty} \gamma_k^2 \ k^{2\alpha},$$

where $(e_k)_{k \in \mathbb{N}}$ is the complete orthonormal basis in $\mathcal{H}$ defined by Assumption 1. We define the operator D^{α} by $D^{\alpha} e_k = k^{\alpha} e_k$, so that $\|u\|_{\alpha} = \|D^{\alpha} u\|$.

Definition 3 *The operator A given by Assumption 1 generates an analytic semigroup $\left\{e^{tA}\right\}_{t \geq 0}$ defined by*

$$e^{At} \left(\sum_{k=1}^{\infty} \gamma_k e_k \right) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \gamma_k e_k, \qquad \forall \ t \geq 0.$$

The analytic semigroup has the following well known properties:

Lemma 4 *Under Assumption 1 there are constants $M > 0$ and $\omega > 0$ such that for all $t > 0$, $\beta \leq \alpha$, and all $u \in \mathcal{H}^{\beta}$*

$$\left\| e^{tA} u \right\|_{\alpha} \leq M t^{-\frac{\alpha - \beta}{m}} \left\| u \right\|_{\beta}, \tag{3}$$

and

$$\left\| e^{tA} P_s u \right\|_{\alpha} \leq M t^{-\frac{\alpha - \beta}{m}} e^{-\omega t} \left\| P_s u \right\|_{\beta}. \tag{4}$$

Assumption 5 (Operator L**)** *Let $L : \mathcal{H}^{\alpha} \to \mathcal{H}^{\alpha - \beta}$ for some $\beta \in [0, m)$ be a continuous linear mapping that in general does not commute with P_c and P_s.*

Assumption 6 (Bilinear Operator B**)** *Let* B *be a bilinear mapping on* $\mathcal{H}^\alpha \times \mathcal{H}^\alpha$ *which is symmetric, i.e.* $B(u,v) = B(v,u)$*. Assume there is a constant* $C > 0$ *such that with* β *defined above*

$$\begin{array}{ll} (B_1) & P_c B(u,u) = 0 \quad \forall u \in \mathcal{N}, \\ (B_2) & \|B(u,v)\|_{\alpha-\beta} \le C\|u\|_\alpha \|v\|_\alpha \quad \forall u,v \in \mathcal{H}^\alpha. \end{array}$$

Denote for shorthand notation $B_s = P_s B$ and $B_c = P_c B$.

For the nonlinearity appearing later in the amplitude equation we define the following.

Definition 7 *Define* $\mathcal{F} : \mathcal{N} \to \mathcal{N}$*, for* $u,\ v,\ w \in \mathcal{N}$*, as*

$$\mathcal{F}(u,v,w) := B_c(u, A_s^{-1} B_s(v,w)). \tag{5}$$

By Assumption 6 the operator $\mathcal{F}$ is continuous, trilinear, and symmetric. One standard example being a cubic like u^3. Moreover, we assume the following:

Assumption 8 (Stability) *Assume that the nonlinearity* $\mathcal{F}$ *satisfies the following conditions, for* $C \in \mathbb{R}$ *and* $c > 0$*.*

$$\langle u, \mathcal{F}(u) \rangle \ge c\,\|u\|^4 - C \quad \forall\ u \in \mathcal{N}, \tag{6}$$

and

$$\langle \mathcal{F}(u+v-w) - \mathcal{F}(u), w \rangle \le c\left[\|v\|^4 + \|u\|^2\|v\|^2 + \|u\|^4\|v\|^2 + \|w\|^2\right], \tag{7}$$

where we define $\mathcal{F}(u) = \mathcal{F}(u,u,u)$ *for short.*

For the noise we suppose:

Assumption 9 (Wiener Process W**)** *Let* W *be a cylindrical Wiener process on* $\mathcal{H}$ *with a bounded covariance operator* Q *defined by* $Qf_k = \alpha_k^2 f_k$ *where* $(\alpha_k)_k$ *is a bounded sequence of real numbers and* $(f_k)_{k\in\mathbb{N}}$ *is an orthonormal basis in* $\mathcal{H}$*. For the orthonormal basis* e_k *from Assumption 1 we assume*

$$\sum_{l=n+1}^{\infty} l^{2\alpha} \lambda_l^{2\gamma-1} \|Q^{\frac{1}{2}} e_l\|^2 < \infty \text{ for some } \gamma \in (0, \tfrac{1}{2})\ . \tag{8}$$

We note that $W(t)$ and $\varepsilon W(\varepsilon^{-2}t)$ are in law the same process due to scaling properties.

Let us discuss two different representations of W. One with the basis e_k and the other one with f_k. For $t \ge 0$, we can write $W(t)$ (cf. Da Prato and Zabczyk [10]) as

$$W(t) := \sum_{k=1}^{\infty} \alpha_k \beta_k(t) f_k = \sum_{l=1}^{\infty} ß_l(t) e_l, \tag{9}$$

where $(\beta_k)_k$ are independent, standard Brownian motions in $\mathbb{R}$. Furthermore, the $ß_l := \sum_{k=1}^{\infty} \alpha_k \langle f_k, e_l \rangle \beta_k$ are real valued Brownian motions, which are in general not independent.

Moreover, it follows easily from the definition of P_c, P_s and $W(t)$ that

$$P_c W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) P_c f_k = \sum_{l=1}^{n} ß_l(t) e_l, \tag{10}$$

and

$$P_sW(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) P_s f_k = \sum_{l=n+1}^{\infty} ß_l(t) e_l, \tag{11}$$

Definition 10 *The stochastic convolution of e^{At} and $W(t)$ is defined by*

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s) = \sum_{l=1}^{\infty} \int_0^t e^{-(t-s)\lambda_l} dß_l(s) e_l \ . \tag{12}$$

For our result we rely on a cut off argument. We consider only solutions (a, ψ) that are not too large. To be more precise we introduce a cut-off time, after which the solution is too big. Later we will show that this time is large with high probability.

Definition 11 ***(Stopping Time)*** *For the $\mathcal{N} \times \mathcal{S}$-valued stochastic process (a, ψ) defined later in (15) we define, for some small $0 < \kappa < \frac{1}{6}$ and some time $T_0 > 0$, the stopping time τ^* as*

$$\tau^* := T_0 \wedge \inf\left\{T > 0 : \|a(T)\|_\alpha > \varepsilon^{-\kappa} \ \textit{or} \ \|\psi(T)\|_\alpha > \varepsilon^{-3\kappa}\right\}. \tag{13}$$

Definition 12 *For a real-valued family of processes $\{X_\varepsilon(t)\}_{t\geq 0}$ we say $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p \geq 1$ there exists a constant C_p such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} |X_\varepsilon(t)|^p \leq C_p f_\varepsilon^p. \tag{14}$$

We use also the analogous notation for time-independent random variables.

Finally note, that we use the letter C for all constants that depend only on other constants like T_0, κ, or α and the data of the equation given by B, W, L, and A.

3 Formal Derivation and the Main Result

Let us first discuss a formal derivation of the Amplitude equation corresponding to Equation (1). We split the solution u into

$$u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon^2 \psi(\varepsilon^2 t) \ , \tag{15}$$

with $a \in \mathcal{N}$ and $\psi \in \mathcal{S}$, and rescale to the slow time scale $T = \varepsilon^2 t$, in order to obtain for the dominant modes

$$da = [L_c a + \varepsilon L_c \psi + 2B_c(a, \psi) + \varepsilon B_c(\psi, \psi)] \, dT + d\widetilde{W}_c. \tag{16}$$

For the fast modes we derive

$$\begin{aligned} d\psi = [&\varepsilon^{-2} A_s \psi + \varepsilon^{-1} L_s a + L_s \psi + \varepsilon^{-2} B_s(a, a) + 2\varepsilon^{-1} B_s(a, \psi) \\ &+ B_s(\psi, \psi)] dT + \varepsilon^{-1} d\widetilde{W}_s \ , \end{aligned} \tag{17}$$

where $\widetilde{W}(T) := \varepsilon W(\varepsilon^{-2} T)$ is a rescaled version of the Wiener process. Now we use (17) in order to remove ψ from Equation (16).

From (17) we obtain in lowest order of ε that

$$A_s\psi \approx -B_s(a,a).$$

As A_s is invertible on $\mathcal{S}$, we derive

$$\psi \approx -A_s^{-1}B_s(a,a),$$

which we substitute into (16). Neglecting all small terms in ε yields

$$da \approx [L_c a - 2\mathcal{F}(a)]\,dT + d\widetilde{W}_c.$$

Thus we consider solutions $b:[0,T_0]\to\mathcal{N}$ of

$$db = [L_c b - 2\mathcal{F}(b)]\,dT + d\widetilde{W}_c. \tag{18}$$

This approximating equation is the amplitude equation that approximates the dynamics of the original SPDE. The main aim of this paper to show that the solution of (1) is

$$u(t) = \varepsilon b(\varepsilon^2 t) + \mathcal{O}(\varepsilon^2)\ .$$

In the following, let us be more precise. Applying Itô's formula to $B_c(a, A_s^{-1}\psi)$ we obtain the amplitude equation with remainder

$$a(T) = a(0) + \int_0^T L_c a(\tau)d\tau - 2\int_0^T \mathcal{F}(a(\tau))d\tau + \widetilde{W}_c(T) + R(T), \tag{19}$$

where the remainder R is given by

$$\begin{aligned}
R(T) = {}& \varepsilon^2 B_c(a(T), A_s^{-1}\psi(T)) - 2\varepsilon^2\int_0^T B_c(B_c(a(\tau),\psi(\tau)), A_s^{-1}\psi(\tau))d\tau\\
&- \varepsilon^3\int_0^T B_c(B_c(\psi(\tau),\psi(\tau)), A_s^{-1}\psi(\tau))d\tau - \varepsilon^2\int_0^T B_c(L_c a, A_s^{-1}\psi)d\tau\\
&- 2\varepsilon\int_0^T B_c(a(\tau), A_s^{-1}B_s(a(\tau),\psi(\tau)))d\tau - \varepsilon^3\int_0^T B_c(L_c\psi, A_s^{-1}\psi)d\tau\\
&- \varepsilon\int_0^T B_c(a, A_s^{-1}L_s a)d\tau - \varepsilon^2\int_0^T B_c(a, A_s^{-1}L_s\psi)d\tau + \varepsilon\int_0^T L_c\psi(\tau)d\tau\\
&- \varepsilon^2\int_0^T B_c(a(\tau), A_s^{-1}B_s(\psi(\tau),\psi(\tau)))d\tau + \varepsilon\int_0^T B_c(\psi(\tau),\psi(\tau))d\tau\\
&- \varepsilon^2\int_0^T B_c(d\widetilde{W}_c(\tau), A_s^{-1}\psi(\tau)) - \varepsilon\int_0^T B_c(a(\tau), A_s^{-1}d\widetilde{W}_s(\tau)).
\end{aligned} \tag{20}$$

For our main aim we need to show that the remainder R is of order ε. This involves carefull analysis of all terms using moments of uniform bounds up to the stopping time like $\mathbb{E}\sup_{[0,\tau^*]}\|R\|_\alpha^p$. Later, we need an explicit error estimate to actually remove R from the equation. Finally, we use the nonlinear stability of the amplitude equation to show that $\tau^* = T_0$ with high probability.

To be more precise, the main result is:

Theorem 13 (Approximation) *Under Assumptions 1, 5, 6 and 9, let u be a solution of (1) defined in (15) with the initial condition $u(0) = \varepsilon a(0) + \varepsilon^2\psi(0)$ where $a(0)$ and $\psi(0)$ are of order one. Suppose that b is a solution of the amplitude equation (18). Then for all $p > 1$ and $T_0 > 0$ there exists $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-7\kappa}\Big) \leq C\varepsilon^p \ . \tag{21}$$

Moreover,

$$\mathbb{P}\left(\tau^* = T_0\right) \geq 1 - C\varepsilon^p \ . \tag{22}$$

The result (22) on the stopping time from Definition 11 essentially tells us that with high probability the solution u for $t \in [0, T_0\varepsilon^{-2}]$ is given by (15) with $a = \mathcal{O}(1)$ and $\psi = \mathcal{O}(1)$. Later in the proof (cf. Lemma 15) we will see that ψ is an Ornstein-Uhlenbeck process plus a quadratic function in a. The quadratic term introduces the cubic in the amplitude equation, while the stochastic part disappears due to averaging effects.

Remark 14 *Let us finally remark without proof, that the scaling assumption on the initial conditions is not very restrictive. Using linear stability the following is easy to show: If $u(0) = \mathcal{O}(\varepsilon)$, then after some time $t_\varepsilon = \mathcal{O}(\ln(1/\varepsilon))$ the following attractivity result holds true*

$$u(t_\varepsilon) = \varepsilon a_\varepsilon + \varepsilon^2\psi_\varepsilon \quad \textit{with } a_\varepsilon, \psi_\varepsilon = \mathcal{O}(1) \ .$$

4 Proof of the Main result

As a first step of the approximation result, we show that in (15) the modes $\psi \in \mathcal{S}$ are essentially an OU-process plus a quadratic term in the modes $a \in \mathcal{N}$. Later we will use this to replace the ψ in (16). After this, we will proceed to show that ψ is with high probability not too large.

Lemma 15 *Under Assumption 1, 5, 6 and 9 let $z(T)$, $T > 0$ be the $\mathcal{S}$-valued process solving the SDE*

$$dz = \varepsilon^{-2}A_s z dT + \varepsilon^{-1}d\widetilde{W}_s, \quad z(0) = \psi(0). \tag{23}$$

Then for $\varepsilon \in (0,1)$ and $T \leq \tau^*$

$$\left\|\psi(T) - z(T) - \varepsilon^{-2}\int_0^T e^{\varepsilon^{-2}A_s(T-\tau)}B_s(a(\tau),a(\tau))d\tau\right\|_\alpha \leq C\varepsilon^{1-5\kappa}. \tag{24}$$

Proof. The mild formulation of (17) is

$$\psi(T) = z(T) + \int_0^T e^{\varepsilon^{-2}A_s(T-\tau)}\left[L_s\psi + \varepsilon^{-1}L_s a + \varepsilon^{-2}B_s(a+\varepsilon\psi)\right]d\tau.$$

Thus we derive

$$
\begin{aligned}
&\Big\|\psi(T)-z(T)-\varepsilon^{-2}\int_0^T e^{\varepsilon^{-2}A_s(T-\tau)}B_s(a,a)d\tau\Big\|_\alpha\\
\le\; &\Big\|\int_0^T e^{\varepsilon^{-2}A_s(T-\tau)}L_s\psi(\tau)d\tau\Big\|_\alpha+\varepsilon^{-1}\Big\|\int_0^T e^{\varepsilon^{-2}A_s(T-\tau)}L_sa(\tau)d\tau\Big\|_\alpha\\
&+2\varepsilon^{-1}\Big\|\int_0^T e^{\varepsilon^{-2}A_s(T-\tau)}B_s(a(\tau),\psi(\tau))d\tau\Big\|_\alpha\\
&+\Big\|\int_0^T e^{\varepsilon^{-2}A_s(T-\tau)}B_s(\psi(\tau),\psi(\tau))d\tau\Big\|_\alpha\\
&=: I_1+I_2+I_3+I_4.
\end{aligned}
$$

We now bound all four terms separately. Using Lemma 4 with $0\le\beta<m$ we obtain for the first term for all $T\le\tau^*$

$$
\begin{aligned}
I_1&=\Big\|\int_0^T e^{\varepsilon^{-2}A_s(T-\tau)}L_s\psi(\tau)d\tau\Big\|_\alpha\\
&\le C\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\,\|\psi(\tau)\|_\alpha\,d\tau\\
&\le C\varepsilon^{2-3\kappa}\,,
\end{aligned}
$$

where we used the definition of τ^* and Assumption 5. Analogously, for the second term, we obtain for all $T\le\tau^*$

$$
I_2\le C\varepsilon^{\frac{2\beta}{m}-1}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\,\|L_sa(\tau)\|_{\alpha-\beta}\,d\tau\le C\varepsilon^{1-\kappa}\,.
$$

For the third term, we obtain

$$
\begin{aligned}
I_3&\le C\varepsilon^{\frac{2\beta}{m}-1}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\|B_s(a(\tau),\psi(\tau))\|_{\alpha-\beta}d\tau\\
&\le C\varepsilon^{\frac{2\beta}{m}-1}\sup_{\tau\in[0,\tau^*]}\|B_s(a(\tau),\psi(\tau))\|_{\alpha-\beta}\cdot\int_0^T e^{-\varepsilon^{-2}\omega\tau}\tau^{-\frac{\beta}{m}}d\tau.
\end{aligned}
$$

Using (B_2) yields for $T\le\tau^*$,

$$
I_3\le C\varepsilon\sup_{\tau\in[0,\tau^*]}\{\|a(\tau)\|_\alpha\|\psi(\tau)\|_\alpha\}\cdot\int_0^{\varepsilon^{-2}\omega T}e^{-\eta}\eta^{-\frac{\beta}{m}}d\eta\le C\varepsilon^{1-4\kappa}.
$$

Analogously, we derive for the fourth term

$$
\begin{aligned}
I_4&\le\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\,\|B_s(\psi(\tau),\psi(\tau))\|_{\alpha-\beta}\,d\tau\\
&\le C\varepsilon^{\frac{2\beta}{m}}\sup_{\tau\in[0,\tau^*]}\|B_s(\psi(\tau),\psi(\tau))\|_{\alpha-\beta}\cdot\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}d\tau\\
&\le C\varepsilon^2\sup_{\tau\in[0,\tau^*]}\|\psi(\tau)\|_\alpha^2\cdot\int_0^{\varepsilon^{-2}\omega T}e^{-\eta}\eta^{-\frac{\beta}{m}}d\eta\le C\varepsilon^{2-6\kappa}\,.
\end{aligned}
$$

Combining all four results yields (24). $\square$

In the following we will show that $\psi \ll \mathcal{O}(\varepsilon^{-3\kappa})$. First, the next Lemma provides bounds for the stochastic convolution based on the well know factorisation method. This also implies bounds for the process z defined in (23).

Lemma 16 *Under Assumption 1, 5 and 9, let $\|z(0)\|_\alpha = \mathcal{O}(1)$. Now for every $\kappa_0 > 0$, $p > 1$ and $T > 0$, there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|z(t)\|_\alpha^{2p}\Big) \le C\varepsilon^{-\kappa_0}. \tag{25}$$

Proof. The mild solution of equation (23) is given by

$$z(t) = e^{\varepsilon^{-2}A_s t} z(0) + \varepsilon^{-1}\widetilde{W}_{\varepsilon^{-2}A_s}(t). \tag{26}$$

The main part in the proof of a bound on $z(t)$ is the bound on $\widetilde{W}_{\varepsilon^{-2}A_s}$. For this, we use the celebrated factorisation method introduced in [11]. Here, for γ from Assumption 9

$$\widetilde{W}_{\varepsilon^{-2}A_s}(t) = C_\gamma \int_0^t e^{\varepsilon^{-2}A_s(t-s)}(t-s)^{\gamma-1}y(s)ds, \tag{27}$$

with $y(s) := \int_0^s e^{\varepsilon^{-2}A_s(s-\sigma)}(s-\sigma)^{-\gamma}d\widetilde{W}_s(\sigma)$. Hence, by Gaussianity

$$\mathbb{E}\,\|y(s)\|_\alpha^{2p} \le C_p\left(\mathbb{E}\,\|y(s)\|_\alpha^2\right)^p$$

Using the series expansion (cf. (11)) yields

$$y(s) = \sum_{l=n+1}^{\infty}\int_0^s e^{-\varepsilon^{-2}(s-\sigma)\lambda_l}(s-\sigma)^{-\gamma}d\widetilde{\beta}_l(\sigma)e_l.$$

From Itô-Isometry

$$\begin{aligned}\mathbb{E}\,\|y(s)\|_\alpha^{2p} &\le C_p\left(\sum_{l=n+1}^{\infty} l^{2\alpha}\mathbb{E}\left(\int_0^s e^{-\varepsilon^{-2}(s-\sigma)\lambda_l}(s-\sigma)^{-\gamma}d\widetilde{\beta}_l(\sigma)\right)^2\right)^p\\ &= C_p\varepsilon^{2p-4p\gamma}\left(\sum_{l=n+1}^{\infty} l^{2\alpha}\left(\lambda_l\right)^{2\gamma-1}\left\|Q^{\frac{1}{2}}e_l\right\|^2\int_0^{\frac{\varepsilon^2 s}{2\lambda_l}} e^{-\tau}\tau^{-2\gamma}d\tau\right)^p,\end{aligned}$$

where we used

$$(d\widetilde{\beta}_l(\sigma))^2 = \sum_{k=1}^{\infty}\alpha_k^2\langle f_k, e_l\rangle^2 d\sigma = \|Q^{\frac{1}{2}}e_l\|^2 d\sigma. \tag{28}$$

Integrating from 0 to T we obtain

$$\mathbb{E}\int_0^T \|y(s)\|_\alpha^{2p}\,ds \le \mathrm{Const}\cdot\varepsilon^{2p-4\gamma p}. \tag{29}$$

Taking the $\mathcal{H}^\alpha$ norm in (27) yields

$$\|\widetilde{W}_{\varepsilon^{-2}A_s}(t)\|_\alpha^{2p} \le C\Big(\int_0^t e^{(-\varepsilon^{-2}\omega)(t-s)}(t-s)^{\gamma-1}\|y(s)\|_\alpha ds\Big)^{2p}.$$

Hölder inequality with $\frac{1}{2p} + \frac{1}{2q} = 1$ for sufficiently large p implies

$$\|\widetilde{W}_{\varepsilon^{-2}A_s}(t)\|_\alpha^{2p} \le \mathrm{Const} \cdot \varepsilon^{4p\gamma-2} \int_0^t \|y(s)\|_\alpha^{2p} ds.$$

Hence, using (29) we obtain

$$\mathbb{E} \sup_{t\in[0,T]} \|\widetilde{W}_{\varepsilon^{-2}A_s}(t)\|_\alpha^{2p} \le C\varepsilon^{4p\gamma-2} \int_0^T \mathbb{E}\|y(s)\|_\alpha^{2p} ds \le C\varepsilon^{2p-2}.$$

For the bound on z take the norm in equation (26) to obtain for sufficiently large p

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} \|z(t)\|_\alpha^{2p} &\le C \left[\mathbb{E} \sup_{t\in[0,T]} \|e^{\varepsilon^{-2}A_s} z(0)\|_\alpha^{2p} + \varepsilon^{-2p}\mathbb{E} \sup_{t\in[0,T]} \|\widetilde{W}_{\varepsilon^{-2}A_s}(t)\|_\alpha^{2p}\right] \\ &\le C\mathbb{E} \sup_{t\in[0,T]} e^{-2p\varepsilon^{-2}\omega t} \|z(0)\|_\alpha^{2p} + C \cdot \varepsilon^{-2p} \cdot \varepsilon^{2p-2} \\ &\le C\varepsilon^{-2}.\end{aligned}$$

Using Hölder inequality we derive for all $p > 1$ and sufficiently large $q > \frac{2}{\kappa_0}$

$$\mathbb{E} \sup_{t\in[0,T]} \|z(t)\|_\alpha^{2p} \le \mathbb{E}\Big(\sup_{t\in[0,T]} \|z(t)\|_\alpha^{2pq}\Big)^{\frac{1}{q}} \le C\varepsilon^{-\kappa_0} ,$$

where the constant C depends among other things on T, p, and κ_0. $\square$

We now need the following simple estimate.

Lemma 17 *Under Assumption 1 and 6, using τ^* defined in Definition 11,*

$$\mathbb{E}\left(\sup_{T\in[0,\tau^*]} \left\| \int_0^T e^{\varepsilon^{-2}A_s(T-\tau)} B_s(a(\tau), a(\tau)) d\tau \right\|_\alpha^{2p}\right) \le C\varepsilon^{4p-4p\kappa}, \qquad (30)$$

for all $\varepsilon \in (0,1)$.

Proof. Using Lemma 4 and (B_2) from Assumption 6 we obtain for $T < \tau^*$

$$\begin{aligned}\left\| \int_0^T e^{\varepsilon^{-2}A_s(T-\tau)} B_s(a,a) d\tau \right\|_\alpha &\le C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)} (T-\tau)^{-\frac{\beta}{m}} \|B_s(a,a)\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} \|a(\tau)\|_\alpha^2 \cdot \int_0^{\varepsilon^{-2}\omega T} e^{-\eta}\eta^{-\frac{\beta}{m}} d\eta \\ &\le C\varepsilon^{2-2\kappa}.\end{aligned}$$

$\square$

Now we can proceed to bound ψ. The following lemma states that $\psi(T)$ is with high probability much smaller than $\varepsilon^{-3\kappa}$, as asserted by the Definition 11 for $T \le \tau^*$. Here a key fact is that in the Definition of τ^* that $a = \mathcal{O}(\varepsilon^{-\kappa})$, while $\psi = \mathcal{O}(\varepsilon^{-3\kappa})$, but we already proved that ψ is essentially a quadratic term in a.

Lemma 18 *Let the assumptions of Lemmas 15, 16, and 17 be true. Then for all $p \geq 1$ there is a constant $C > 0$ such that*

$$\mathbb{E} \sup_{T\in[0,\tau^*]} \|\psi(T)\|_\alpha^{2p} \leq C\varepsilon^{-4p\kappa}. \tag{31}$$

Proof. From (24), by triangle inequality and Lemma 15, we obtain

$$\mathbb{E} \sup_{[0,\tau^*]} \|\psi\|_\alpha^{2p} \leq C\varepsilon^{2p-10p\kappa} + C\mathbb{E} \sup_{[0,\tau^*]} \|z\|_\alpha^{2p} + C\varepsilon^{-4p}\mathbb{E} \sup_{[0,\tau^*]} \Big\| \int_0^T e^{\varepsilon^{-2} A_s(T-\tau)} B_s(a,a) d\tau \Big\|_\alpha^{2p} .$$

Using Lemma 16 and 17 we finish the proof. □

Corollary 19 *Under the assumptions of Lemma* 18*, there is for every every $p > 1$ a constant $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{T\in[0,\tau^*]} \|\psi(T)\|_\alpha < \varepsilon^{-3\kappa} \Big) \geq 1 - C\varepsilon^{2p\kappa}. \tag{32}$$

Proof. From Chebychev inequality

$$\mathbb{P}\Big(\sup_{[0,\tau^*]} \|\psi\|_\alpha < \varepsilon^{-3\kappa} \Big) \geq 1 - \varepsilon^{6\kappa p} \cdot \mathbb{E} \sup_{[0,\tau^*]} \|\psi\|_\alpha^{2p}.$$

We finish the proof by using (31). □

Now the next step is to bound the remainder R defined in (20), and use it in order to show the approximation result later.

Lemma 20 *We assume that Assumptions 1, 5, 6, and 9 hold. Then for all $p > 1$ there exists a constant $C > 0$ such that*

$$\mathbb{E} \sup_{T\in[0,\tau^*]} \|R(T)\|_\alpha^{p} \leq C\varepsilon^{p-6p\kappa}. \tag{33}$$

Proof. For the bound on R we bound all terms in (20) separately. The estimates rely on Condition (B_2) and the inequality $\|\psi\|_\gamma \leq C\|\psi\|_{\gamma+\delta}$ for all $\gamma \in \mathbb{R}$ and $\delta \geq 0$. Moreover, we use that $B_c(a(\tau), A_s^{-1}\psi(\tau)) \in \mathcal{N}$ (finite dimensional) and A_s^{-1} being a bounded linear operator on $\mathcal{S} \subset \mathcal{H}^\alpha$ to obtain for all times up to the stopping time τ^* that

$$\begin{aligned}\left\|\varepsilon^2 B_c(a, A_s^{-1}\psi)\right\|_\alpha &\leq C\varepsilon^2 \left\|B_c(a, A_s^{-1}\psi)\right\|_{\alpha-\beta} \leq C\varepsilon^2 \left\|a\right\|_\alpha \left\|A_s^{-1}\psi\right\|_\alpha \\ &\leq C\varepsilon^2 \left\|a\right\|_\alpha \left\|\psi\right\|_\alpha .\end{aligned}$$

Using the definition of τ^*, we obtain

$$\mathbb{E} \sup_{[0,\tau^*]} \|\varepsilon^2 B_c(a, A_s^{-1}\psi)\|_\alpha^p \leq C\varepsilon^{2p-4p\kappa}. \tag{34}$$

For the second term in (20) with $T \leq \tau^* \leq T_0$

$$\begin{aligned}\Big\|2\varepsilon^2 \int_0^T B_c(B_c(a,\psi), A_s^{-1}\psi) d\tau\Big\|_\alpha &\leq C\varepsilon^2 \int_0^T \|B_c(B_c(a,\psi), A_s^{-1}\psi)\|_{\alpha-\beta} d\tau \\ &\leq C\varepsilon^2 T \cdot \sup_{[0,\tau^*]} \|B_c(a,\psi)\|_\alpha \|A_s^{-1}\psi\|_\alpha \\ &\leq C\varepsilon^2 T \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|\psi\|_\alpha^2 \\ &\leq C\varepsilon^{2-7\kappa}. \end{aligned} \tag{35}$$

Analogously, for the third term in (20)

$$\left\|\varepsilon^3 \int_0^T B_c(B_c(\psi,\psi), A_s^{-1}\psi)d\tau\right\|_\alpha \le C\varepsilon^3 \int_0^T \|B_c(B_c(\psi,\psi), A_s^{-1}\psi)\|_{\alpha-\beta} d\tau$$
$$\le C\varepsilon^3 T \cdot \sup_{[0,\tau^*]} \|\psi\|_\alpha^3 \le C\varepsilon^{3-9\kappa}. \tag{36}$$

The 4th term in (20) is bounded by

$$\begin{aligned}\left\|\varepsilon^2 \int_0^T B_c(L_c a, A_s^{-1}\psi)d\tau\right\|_\alpha &\le C\varepsilon^2 \int_0^T \|B_c(L_c a, A_s^{-1}\psi)\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon^2 \cdot \sup_{[0,\tau^*]} \|L_c a\|_\alpha \|A_s^{-1}\psi\|_\alpha \\ &\le C\varepsilon^2 \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|\psi\|_\alpha \\ &\le C\varepsilon^{2-4\kappa},\end{aligned} \tag{37}$$

where we used $\|L_c a\|_\alpha \le C\|L_c a\|_{\alpha-\beta}$, as $\mathcal{N}$ is finite dimensional.

For the 5th term in (20)

$$\begin{aligned}\left\|2\varepsilon \int_0^T B_c(a, A_s^{-1}B_s(a,\psi))d\tau\right\|_\alpha &\le C\varepsilon \int_0^T \|B_c(a, A_s^{-1}B_s(a,\psi))\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|A_s^{-1}B_s(a,\psi)\|_\alpha \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha^2 \|\psi\|_\alpha \\ &\le C\varepsilon^{1-5\kappa}.\end{aligned} \tag{38}$$

The 6th term in (20) is bounded by

$$\begin{aligned}\left\|\varepsilon^3 \int_0^T B_c(L_c\psi, A_s^{-1}\psi)d\tau\right\|_\alpha &\le C\varepsilon^3 \int_0^T \|B_c(L_c\psi(\tau), A_s^{-1}\psi(\tau))\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon^3 \cdot \sup_{[0,\tau^*]} \|L_c\psi\|_\alpha \|A_s^{-1}\psi)\|_\alpha \\ &\le C\varepsilon^3 \cdot \sup_{[0,\tau^*]} \|\psi\|_\alpha^2 \\ &\le C\varepsilon^{3-6\kappa}.\end{aligned} \tag{39}$$

The 7th term in (20) is bounded by

$$\begin{aligned}\left\|\varepsilon \int_0^T B_c(a, A_s^{-1}L_s a)d\tau\right\|_\alpha &\le C\varepsilon \int_0^T \|B_c(a, A_s^{-1}L_s a)\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|A_s^{-1}L_s a\|_\alpha \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha \|L_s a\|_{\alpha-m} \\ &\le C\varepsilon \cdot \sup_{[0,\tau^*]} \|a\|_\alpha^2 \\ &\le C\varepsilon^{1-2\kappa}.\end{aligned} \tag{40}$$

The 8th term in (20) is completely analogous. We have

$$\Big\|\varepsilon^2\int_0^T B_c(a,A_s^{-1}L_s\psi)d\tau\Big\|_\alpha \le C\varepsilon^{2-4\kappa}. \tag{41}$$

Moreover for the 9th term in (20):

$$\Big\|\varepsilon\int_0^T B_c(\psi,\psi)d\tau\Big\|_\alpha \le C\varepsilon\int_0^T \|B_c(\psi,\psi)\|_{\alpha-\beta}d\tau \le C\varepsilon^{1-6\kappa}\,. \tag{42}$$

For the 10th term in (20)

$$\begin{aligned}\Big\|\varepsilon\int_0^T L_c\psi d\tau\Big\|_\alpha &\le C\varepsilon\int_0^T \|L_c\psi\|_\alpha d\tau \le C\varepsilon\int_0^T \|L_c\psi\|_{\alpha-\beta}d\tau\\ &\le C\varepsilon\cdot\sup_{[0,\tau^*]}\|\psi(\tau)\|_\alpha \le C\varepsilon^{1-3\kappa}.\end{aligned} \tag{43}$$

The 11th term in (20) is bounded by

$$\begin{aligned}\Big\|\varepsilon^2\int_0^T B_c(a,A_s^{-1}B_s(\psi,\psi))d\tau\Big\|_\alpha &\le C\varepsilon^2\int_0^T \|B_c(a,A_s^{-1}B_s(\psi,\psi))\|_{\alpha-\beta}d\tau\\ &\le C\varepsilon^2\cdot\sup_{[0,\tau^*]}\|a\|_\alpha\|A_s^{-1}B_s(\psi,\psi)\|_\alpha\\ &\le C\varepsilon^2\sup_{[0,\tau^*]}\|a\|_\alpha\|\psi\|_\alpha^2\\ &\le C\varepsilon^{2-7\kappa}.\end{aligned} \tag{44}$$

For the stochastic integral $\varepsilon^2\int_0^T B_c(d\widetilde{W}_c,A_s^{-1}\psi)$ in (20) note that the covariance operator of W_c is $Q_c=P_cQP_c$. Define

$$\pounds(\tau)u := B_c(u(\tau),A_s^{-1}\psi(\tau)),$$

to obtain

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T B_c(d\widetilde{W}_c(\tau),A_s^{-1}\psi(\tau))\Big\|_\alpha^p = \mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \pounds(\tau)d\widetilde{W}_c(\tau)\Big\|_\alpha^p.$$

By Burkholder-Davis-Gundy (cf. Theorem 1.2.4 in [18]) we derive

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \pounds d\widetilde{W}_c\Big\|_\alpha^p &= \mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T D^\alpha\pounds d\widetilde{W}_c\Big\|^p\\ &\le C\cdot\mathbb{E}\Big(\int_0^{\tau^*}\|D^\alpha\pounds Q_c^{\frac12}\|_{HS}^2d\tau\Big)^{\frac{p}{2}}\\ &= C\cdot\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty\|D^\alpha\pounds Q_c^{\frac12}g_k\|^2d\tau\Big)^{\frac{p}{2}},\end{aligned}$$

where $(g_k)_{k\in\mathbb{N}}$ is any orthonormal basis in $\mathcal{H}$ and D^α was defined in Definition 2. The space HS is the space of Hilbert-Schmidt operators on $\mathcal{H}$, equipped with the norm $\|\Psi\|_{HS} = \mathrm{Trace}[\Psi\Psi^*]$. Hence,

$$
\begin{aligned}
\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \pounds d\widetilde{W}_c\Big\|_\alpha^p &\le C\cdot\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty \|D^\alpha B_c(Q_c^{\frac12}g_k, A_s^{-1}\psi)\|^2 d\tau\Big)^{\frac{p}{2}} \\
&= C\cdot\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty \|\underbrace{B_c(Q_c^{\frac12}g_k, A_s^{-1}\psi)}_{\in\mathcal{N}}\|_\alpha^2 d\tau\Big)^{\frac{p}{2}} \\
&\le C\mathbb{E}\Big(\sum_{k=1}^\infty \sup_{[0,\tau^*]}\|B_c(Q_c^{\frac12}g_k, A_s^{-1}\psi)\|_{\alpha-\beta}^2\Big)^{\frac{p}{2}} \\
&\le C\Big(\sum_{k=1}^\infty \|Q_c^{\frac12}g_k\|_\alpha^2\Big)^{\frac{p}{2}}\cdot\mathbb{E}\sup_{[0,\tau^*]}\|A_s^{-1}\psi(\tau)\|_\alpha^p \\
&\le C\varepsilon^{-3p\kappa}, \qquad (45)
\end{aligned}
$$

where we used the fact that the norm in HS is invariant under taking the adjoint, and independent of the choice of the basis, in order to obtain

$$
\begin{aligned}
\sum_{k=1}^\infty \|Q_c^{\frac12}g_k\|_\alpha^2 = \|D^\alpha Q_c^{\frac12}\|_{HS}^2 = \|Q_c^{\frac12}D^\alpha\|_{HS}^2 &= \sum_{k=1}^\infty \|Q_c^{\frac12}D^\alpha e_k\|^2 \\
&= \sum_{k=1}^\infty \langle Q_c^{\frac12}D^\alpha e_k, Q_c^{\frac12}D^\alpha e_k\rangle = \sum_{k=1}^\infty k^{2\alpha}\langle P_cQP_ce_k, e_k\rangle \\
&= \sum_{k=1}^n k^{2\alpha}\langle Qe_k, e_k\rangle = \sum_{k=1}^n k^{2\alpha}\|Q^{\frac12}e_k\|^2 \le C\,.
\end{aligned}
$$

For $\varepsilon\int_0^T B_c(a, A_s^{-1}d\widetilde{W}_s)$, the last stochastic integral in (20), note that the covariance operator of $\widetilde{W}_s$ is $Q_s = P_sQP_s$. Similar to the previous estimate we define

$$\pounds_1(\tau)u := B_c(a(\tau), A_s^{-1}u).$$

Now by Burkholder-Davis-Gundy (cf. Theorem 1.2.4 in [18]) we obtain

$$
\begin{aligned}
\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T B_c(a, A_s^{-1}d\widetilde{W}_s)\Big\|_\alpha^p &= \mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T D^\alpha\pounds_1 d\widetilde{W}_s\Big\|^p \\
&= C\cdot\mathbb{E}\Big(\varepsilon^2\int_0^{\tau^*}\|D^\alpha\pounds_1Q_s^{\frac12}\|_{HS}^2 d\tau\Big)^{\frac{p}{2}} \\
&= C\cdot\mathbb{E}\Big(\varepsilon^2\int_0^{\tau^*}\sum_{k=1}^\infty\|D^\alpha\pounds_1Q_s^{\frac12}e_k\|^2 d\tau\Big)^{\frac{p}{2}} \\
&= C\varepsilon^p\cdot\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty\|D^\alpha B_c(a, A_s^{-1}Q_s^{\frac12}e_k)\|^2 d\tau\Big)^{\frac{p}{2}} \\
&\le C\varepsilon^p\cdot\mathbb{E}\Big(\sum_{k=1}^\infty\sup_{[0,\tau^*]}\|B_c(a, A_s^{-1}Q_s^{\frac12}e_k)\|_{\alpha-\beta}^2\Big)^{\frac{p}{2}} \\
&\le C\varepsilon^{p-p\kappa}\Big(\sum_{k=1}^\infty\|A_s^{-1}Q_s^{\frac12}e_k\|_\alpha^2\Big)^{\frac{p}{2}} \\
&\le C\varepsilon^{p-p\kappa}, \qquad (46)
\end{aligned}
$$

where we used

$$
\begin{aligned}
\sum_{k=1}^{\infty}\|A_s^{-1}Q_s^{\frac{1}{2}}e_k\|_\alpha^2 = \|D^\alpha A_s^{-1}Q_s^{\frac{1}{2}}\|_{HS}^2 = \|Q_s^{\frac{1}{2}}A_s^{-1}D^\alpha\|_{HS}^2 = \sum_{k=1}^{\infty}\|Q_s^{\frac{1}{2}}A_s^{-1}D^\alpha e_k\|^2 \\
= \sum_{k=1}^{\infty}\frac{k^{2\alpha}}{\lambda_k^2}\|Q_s^{\frac{1}{2}}e_k\|^2 = \sum_{k=1}^{\infty}\frac{k^{2\alpha}}{\lambda_k^2}\langle P_sQP_se_k, e_k\rangle \\
= \sum_{k=n+1}^{\infty}\frac{k^{2\alpha}}{\lambda_k^2}\|Q^{\frac{1}{2}}e_k\|^2 \le C\ .
\end{aligned}
$$

The last step follows from Assumption 9, as $\lambda_k \to \infty$.

As we supposed $\kappa < \frac{1}{6}$ in the definition of τ^*, we can collect all term in the equations from (34) until (46). This implies the result. □

In order to prove now the approximation result, we first need the following a-priori estimate for solutions of the amplitude equation.

Lemma 21 *Let Assumptions 1, 5, 8 and 9 hold. Define the stochastic process $b(T)$ in $\mathcal{N}$ as the solution of*

$$
b(T) = b(0) + \int_0^T L_c b(\tau)d\tau - 2\int_0^T \mathcal{F}(b(\tau))d\tau + \widetilde{W}_c(T). \tag{47}
$$

If, for some $p > \frac{1}{2}$, the initial condition satisfies $\mathbb{E}\|b(0)\|^{4p-2} \le C$ and $\mathbb{E}\|b(0)\|^{2p} \le C$, then for all $T_0 > 0$ there exists another constant $C > 0$ such that

$$
\mathbb{E}\sup_{T\in[0,T_0]}\|b(T)\|_\alpha^p \le C\ . \tag{48}
$$

We note that all norms in a finite dimensional space are equivalent. Thus for simplicity of notation in the proof we use only the standard Eucledian norm and suppose that $b \in \mathbb{R}^n$.

Proof. The existence and uniqueness of solutions for equation (47) is standard. To verify the bound in (48) we will first show that there is a T-independent constant such that

$$
\mathbb{E}\,\|b(T)\|^{2p} \le Const \quad \forall\, T > 0\ . \tag{49}
$$

Apply Itô's formula to $\|b(T)\|^{2p}$ to derive

$$
\begin{aligned}
\|b(T)\|^{2p} &= \|b(0)\|^{2p} + 2p\int_0^T \|b(s)\|^{2(p-1)}\langle b(s), db(s)\rangle \\
&\quad + p\int_0^T \|b(s)\|^{2(p-1)}\langle db(s), db(s)\rangle + 2p(p-1)\int_0^T \|b(s)\|^{2(p-2)}\langle b(s), db(s)\rangle^2\ .
\end{aligned}
$$

From (47), we obtain

$$
\begin{aligned}
\|b(T)\|^{2p} &= \|b(0)\|^{2p} + 2p\int_0^T \|b(s)\|^{2(p-1)}\langle b(s), L_c b(s)\rangle\, ds \\
&\quad - 4p\int_0^T \|b(s)\|^{2(p-1)}\langle b(s), \mathcal{F}(b(s))\rangle ds + 2p\int_0^T \|b(s)\|^{2(p-1)}\langle b(s), d\widetilde{W}_c\rangle \\
&\quad + p\int_0^T \|b(s)\|^{2(p-1)}\langle d\widetilde{W}_c, d\widetilde{W}_c\rangle + 2p(p-1)\int_0^T \|b(s)\|^{2(p-2)}\langle b(s), d\widetilde{W}_c\rangle^2.
\end{aligned}
$$

Thus

$$\begin{aligned}\|b(T)\|^{2p} \le \|b(0)\|^{2p} + C\int_0^T \|b(s)\|^{2(p-1)} ds + c\int_0^T \|b(s)\|^{2p} ds \\ - 4p\int_0^T \|b(s)\|^{2(p-1)}\langle b(s), \mathcal{F}(b(s))\rangle ds \\ + 2p\int_0^T \|b(s)\|^{2(p-1)}\langle b(s), d\widetilde{W}_c\rangle, \end{aligned} \tag{50}$$

where we used

$$\langle b(s), L_c b(s)\rangle \overset{\text{C.S}}{\le} \|b(s)\| \, \|L_c b(s)\| \le C\,\|b(s)\|^2,$$

and

$$\begin{aligned}\langle d\widetilde{W}_c, d\widetilde{W}_c\rangle = \sum_{k=1}^n \sum_{l=1}^n d\text{ß}_k d\text{ß}_l \underbrace{\langle e_k, e_l\rangle}_{=0 \text{ if } k\neq l} &= \sum_{k=1}^n \sum_{l=1}^n \alpha_l^2 \langle f_l, e_k\rangle^2 dt \\ &= \sum_{l=1}^n \|Q^{\frac{1}{2}} e_l\|^2 dt \underset{\text{bdd}}{\overset{Q \text{ is}}{\le}} C dt,\end{aligned}$$

together with

$$\begin{aligned}\langle b, d\widetilde{W}_c\rangle\langle b, d\widetilde{W}_c\rangle &= \sum_{k=1}^\infty \sum_{l=1}^\infty \alpha_k\alpha_l \underbrace{d\beta_k d\beta_l}_{=0 \quad \text{if } k\neq l} \langle b, f_k\rangle\langle b, f_l\rangle = \sum_{k=1}^\infty \alpha_k^2 \langle b, f_k\rangle^2 dt \\ &= \sum_{k=1}^\infty \langle Q^{\frac{1}{2}} b, f_k\rangle^2 dt = \left\|Q^{\frac{1}{2}} b\right\|^2 dt \underset{\text{bdd}}{\overset{Q \text{ is}}{\le}} C\|b\|^2 dt.\end{aligned}$$

Using (6), we simplify (50) to

$$\begin{aligned}\|b(T)\|^{2p} \le \|b(0)\|^{2p} + C\int_0^T \|b(s)\|^{2p-2} ds + c\int_0^T \|b(s)\|^{2p} ds \\ - c\int_0^T \|b(s)\|^{2p+2} ds + 2p\int_0^T \|b(s)\|^{2(p-1)} \langle b(s), d\widetilde{W}_c\rangle.\end{aligned}$$

If we use for any $\delta > 0$ the inequality $\|b(T)\|^q \le \delta\|b(T)\|^{2p+2} + C_{\delta,q,p}$ for $q \in (0, 2p+2)$, then we obtain

$$\begin{aligned}\|b(T)\|^{2p} \le \|b(0)\|^{2p} + C_{\delta,q,p}T - C\int_0^T \|b(s)\|^{2p+2} ds \\ + 2p\int_0^T \|b(s)\|^{2(p-1)}\langle b(s), d\widetilde{W}_c\rangle.\end{aligned} \tag{51}$$

Taking expectations on both sides yields

$$\begin{aligned}\mathbb{E}\,\|b(T)\|^{2p} &\le \mathbb{E}\,\|b(0)\|^{2p} + C_{\delta,q,p} - C\mathbb{E}\int_0^T \|b(s)\|^{2p+2} ds \\ &\le \mathbb{E}\,\|b(0)\|^{2p} + C_{\delta,q,p},\end{aligned}$$

where we used $\mathbb{E}\int_0^T \|b(s)\|^{2(p-1)} \langle b(s), d\widetilde{W}_c\rangle = 0$. This finishes, the proof of the first part.

For the second part, we take first supremum and then expectation in Equation (51), and end up with

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\|b(T)\|^{2p} &\leq \mathbb{E}\|b(0)\|^{2p} + C_{\delta,q,p}T_0 \\ &\quad + 2p\mathbb{E}\sup_{T\in[0,T_0]}\int_0^T \|b(s)\|^{2(p-1)}\langle b(s), d\widetilde{W}_c\rangle.\end{aligned}$$

From Burkholder-Davis-Gundy inequality

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\|b(T)\|^{2p} &\leq \mathbb{E}\|b(0)\|^{2p} + C_{\delta,q,p}T_0 + 2p\mathbb{E}\Big(\int_0^{T_0} \|b(s)\|^{4(p-1)}\langle b(s), d\widetilde{W}_c\rangle^2\Big)^{\frac{1}{2}} \\ &\leq C + 2p\mathbb{E}\Big(\int_0^{T_0} \|b(s)\|^{4p-2} ds\Big)^{\frac{1}{2}} \leq C\ ,\end{aligned}$$

where we used Cauchy-Schwarz inequality and our first bound (cf. (49)) on b with $2p-1$ instead of p. This finishes the proof. $\square$

Together with the apriori bounds on solutions of the amplitude equation and the bounds on the remainder R, we are now ready to prove the main approximation result together with an a-priori bound on a.

Theorem 22 *We assume that Assumption 1, 5, 8, 6 and 9 hold. Let b be a solution of (47) and a as defined in (19). If the initial conditions satisfies $a(0) = b(0)$ and $\mathbb{E}\|a(0)\|^p \leq C_p$, for all $p > 1$, then*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\|a(T) - b(T)\|_\alpha^p \leq C\varepsilon^{p-6p\kappa}, \tag{52}$$

and

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\|a(T)\|_\alpha^p \leq C. \tag{53}$$

Proof. Define $\varphi(T)$ as

$$\varphi(T) := a(T) - R(T).$$

From (19) we obtain

$$\varphi(T) = a(0) + \int_0^T L_c\left[\varphi(\tau) + R(\tau)\right]d\tau - 2\int_0^T \mathcal{F}(\varphi(\tau) + R(\tau))d\tau + \widetilde{W}_c(T). \tag{54}$$

Define now $h(T)$ by

$$h(T) := b(T) - \varphi(T).$$

Subtracting (54) from (47), we obtain

$$h(T) = \int_0^T L_c h(\tau)d\tau - \int_0^T L_c R(\tau)d\tau + 2\int_0^T [\mathcal{F}(b - h + R) - \mathcal{F}(b)](\tau)d\tau.$$

Thus

$$\partial_T h = L_c h - L_c R + 2[\mathcal{F}(b - h + R) - \mathcal{F}(b)]\ . \tag{55}$$

Taking the scalar product $\langle \cdot, h\rangle$ on both sides of (55) yields

$$\tfrac{1}{2}\partial_T \|h\|^2 = \langle \partial_T h, h\rangle = \langle L_c h, h\rangle - \langle L_c R, h\rangle + 2\langle \mathcal{F}(b-h+R) - \mathcal{F}(b), h\rangle \,.$$

Using Cauchy-Schwarz inequality and (7) , we obtain the following linear ordinary differential inequality

$$\partial_T \|h\|^2 \leq c\,\|h\|^2 + c\left[\|R\|^4 + \|b\|^2\|R\|^2 + \|b\|^4\|R\|^2\right].$$

We apply now a comparison argument to deduce

$$\|h(T)\|^2 \leq c\int_0^T e^{c(T-\tau)}\left[\|R\|^4 + \|b\|^2\|R\|^2 + \|b\|^4\|R\|^2\right]d\tau.$$

Thus for $T \leq \tau^*$

$$\begin{aligned}\|h(T)\|^2 &\leq c\left(e^{cT}-1\right)\sup_{[0,\tau^*]}\left[\|R\|^4 + \|b\|^2\|R\|^2 + \|b\|^4\|R\|^2\right]\\ &\leq ce^{cT}\sup_{[0,\tau^*]}\|R\|^2\left[\|R\|^2 + \|b\|^2 + \|b\|^4\right].\end{aligned}$$

Using Young inequality, we obtain $\|b\|^2 \leq \frac{1}{2}\|b\|^4 + \frac{1}{2}$, and hence

$$\|h(T)\|^2 \leq ce^{cT_0}\sup_{[0,\tau^*]}\|R\|^2\Big[\tfrac{1}{2} + \sup_{[0,\tau^*]}\|R\|^2 + \tfrac{3}{2}\sup_{[0,\tau^*]}\|b\|^4\Big]\,.$$

Now from Lemmas 20 and 21, we obtain

$$\begin{aligned}\mathbb{E}\sup_{[0,\tau^*]}\|h\|^{2p} &\leq C\Big(\mathbb{E}\sup_{[0,\tau^*]}\|R\|^{4p}\Big)^{\frac{1}{2}}\cdot\Big[\mathbb{E}\Big(\tfrac{1}{2} + \sup_{[0,\tau^*]}\|R\|^2 + \tfrac{3}{2}\sup_{[0,\tau^*]}\|b\|^4\Big)^{2p}\Big]^{\frac{1}{2}}\\ &\leq C\varepsilon^{2p(1-6\kappa)}\cdot\Big[C_1 + C_2\varepsilon^{2p(1-6\kappa)}\Big]^{\frac{1}{2}}\,.\end{aligned}$$

Thus

$$\mathbb{E}\sup_{[0,\tau^*]}\|h\|^{2p} \leq C\varepsilon^{2p-12p\kappa}, \tag{56}$$

where we used $\left[C_1 + C_2\varepsilon^{2p(1-6\kappa)}\right]^{\frac{1}{2}} \leq C$ for $\varepsilon \in (0,1)$ in case $\kappa < \frac{1}{6}$. Thus

$$\begin{aligned}\mathbb{E}\sup_{[0,\tau^*]}\|a-b\|^{2p} &= \mathbb{E}\sup_{[0,\tau^*]}\|h-R\|^{2p}\\ &\leq 2p\Big[\mathbb{E}\sup_{[0,\tau^*]}\|h\|^{2p} + \mathbb{E}\sup_{[0,\tau^*]}\|R\|^{2p}\Big].\end{aligned}$$

Using Lemma 20 and (56), this immediately finishes the first part.

For the second part of the theorem we consider

$$\mathbb{E}\sup_{[0,\tau^*]}\|a\|^{2p} \leq 2p\mathbb{E}\sup_{[0,\tau^*]}\|a-b\|^{2p} + 2p\mathbb{E}\sup_{[0,\tau^*]}\|b\|^{2p}\,.$$

Using the first part and (48), we obtain (53) for $\varepsilon \in (0,1)$ and $\kappa < \frac{1}{6}$. $\square$

Corollary 23 *Under the assumptions of Theorem 22, we obtain, for every $p \geq 1$,*

$$\mathbb{P}\Big(\sup_{T\in[0,\tau^*]} \|a(T)\|_\alpha < \varepsilon^{-\kappa} \Big) \geq 1 - C\varepsilon^{2\kappa p}. \tag{57}$$

Proof. Chebychev inequality implies

$$\mathbb{P}\Big(\sup_{[0,\tau^*]} \|a\|_\alpha < \varepsilon^{-\kappa} \Big) \geq 1 - \varepsilon^{2\kappa p}\mathbb{E}\Big(\sup_{[0,\tau^*]} \|a\|_\alpha^{2p} \Big) .$$

Using (52), this immediately yields the result. □

Finally, we use the results previously obtained to prove the main result of Theorem 13 for the approximation of the solution of the SPDE (1).

Proof of Theorem 13. First we verify the result for the stopping time. For this note that

$$\begin{aligned} \mathbb{P}\Big(\tau^* = T_0\Big) &\geq \mathbb{P}\Big(\sup_{T\in[0,\tau^*]} \|a(T)\|_\alpha < \varepsilon^{-\kappa}, \sup_{T\in[0,\tau^*]} \|\psi(T)\|_\alpha < \varepsilon^{-3\kappa} \Big) \\ &\geq 1 - \mathbb{P}\Big(\sup_{T\in[0,\tau^*]} \|a(T)\|_\alpha \geq \varepsilon^{-\kappa} \Big) - \mathbb{P}\Big(\sup_{T\in[0,\tau^*]} \|\psi(T)\|_\alpha \geq \varepsilon^{-3\kappa} \Big) \end{aligned}$$

Using Lemma 18 and Theorem 22 we obtain (22).

Now let us turn to the approximation result. Using (15) and triangle inequality, we obtain

$$\mathbb{E} \sup_{T\in[0,\tau^*]} \|u(\varepsilon^{-2}T) - \varepsilon b(T)\|_\alpha^p \leq C\Big[\varepsilon^p \mathbb{E} \sup_{[0,\tau^*]} \|a-b\|_\alpha^p + \varepsilon^{2p}\mathbb{E} \sup_{[0,\tau^*]} \|\psi\|_\alpha^p\Big].$$

From (31) and (52), we obtain for all $q > 0$

$$\mathbb{E} \sup_{t\in[0,\varepsilon^{-2}\tau^*]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha^q \leq C\varepsilon^{2q-6q\kappa} . \tag{58}$$

As

$$\begin{aligned} &\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-7\kappa} \Big) \\ &\quad = \mathbb{P}\Big(\sup_{T\in[0,T_0]} \|u(\varepsilon^{-2}T) - \varepsilon b(T)\|_\alpha > \varepsilon^{2-7\kappa}, \ \tau^* = T_0 \Big) \\ &\quad\quad + \mathbb{P}\Big(\sup_{T\in[0,T_0]} \|u(\varepsilon^{-2}T) - \varepsilon b(T)\|_\alpha > \varepsilon^{2-7\kappa}, \ \tau^* < T_0 \Big) \\ &\quad \leq \mathbb{P}\Big(\sup_{T\in[0,\tau^*]} \|u(\varepsilon^{-2}T) - \varepsilon b(T)\|_\alpha > \varepsilon^{2-7\kappa} \Big) + \mathbb{P}(\tau^* < T_0) \end{aligned}$$

Using Chebychev inequality and (22), yields for all $q > 0$

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-7\kappa} \Big) \leq C\varepsilon^p + \frac{1}{\varepsilon^{2q-7q\kappa}}\mathbb{E} \sup_{t\in[0,\varepsilon^{-2}\tau^*]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha^q ,$$

and we finish the proof by using (58) and putting $q = \frac{p}{\kappa}$. □

5 Application

There are numerous examples in the physics literature of equations with quadratic nonlinearities where our theory does apply. Before we give examples, we suppose in our applications for simplicity that W is a cylindrical Wiener process on $\mathcal{H}$ with a covariance operator Q defined by $Qe_k = \alpha_k^2 e_k$ where $(\alpha_k)_k$ is a bounded sequence of real numbers and e_k are the eigenfunctions of the dominant linear operator.

5.1 Burgers equation

One example is the Burgers equation (cf. (2)) on the interval $[0,\pi]$, with Dirichlet boundary conditions. We take

$$\mathcal{H} = L^2([0,\pi]), \quad e_k(x) = \sqrt{\tfrac{2}{\pi}}\sin(kx) \quad \text{and} \quad \mathcal{N} = \text{span}\{\sin\}.$$

We note that the Assumption 1 is true, where the eigenvalues of $-A = -\partial_x^2 - 1$ are $\lambda_k = k^2 - 1$ with $m = 2$ and $\lim_{k\to\infty}\lambda_k = \infty$. If we fix P_c to be the $\mathcal{H}$-orthogonal projection onto $\mathcal{N}$, then both P_c and P_s commute with A.

Moreover, all conditions of Assumption 6 are satisfied with

$$B(u,v) = \tfrac{1}{2}\partial_x(uv).$$

Condition (B_1) is true, as for $u = \gamma \sin \in \mathcal{N}$

$$P_c B(u,u) = P_c\left[\gamma^2 \sin(x)\cos(x)\right] = 0.$$

For Condition (B_2) with $\alpha = \frac{1}{4}$ and $\beta = \frac{5}{4} < m$, we verify using Sobolev embedding from $\mathcal{H}^{1/4}$ into L^4

$$\begin{aligned} 2\|B(u,v)\|_{\mathcal{H}^{-1}} &= \|\partial_x(uv)\|_{\mathcal{H}^{-1}} \le \|uv\|_{L^2} \\ &\le C\|u\|_{L^4}\|v\|_{L^4} \le C\|u\|_{\mathcal{H}^{\frac{1}{4}}}\|v\|_{\mathcal{H}^{\frac{1}{4}}}. \end{aligned}$$

We derive after a straightforward calculation that

$$\mathcal{F}(\gamma\sin) = \frac{1}{12}\gamma^3\sin.$$

This function is trilinear, continuous and satisfies the conditions (6) and (7), where

$$\langle \gamma_1\sin, \mathcal{F}(\gamma_1\sin)\rangle = c\gamma_1^4,$$

and

$$\begin{aligned} &\langle \mathcal{F}(\gamma_1\sin + \gamma_2\sin - \gamma_3\sin) - \mathcal{F}(\gamma_1\sin), \gamma_3\sin\rangle \\ &= c\{\underbrace{\gamma_2^3\gamma_3 + 3\gamma_2\gamma_3^3}_{:=I_1} + \underbrace{3\gamma_1\gamma_2^2\gamma_3 - 3\gamma_2^2\gamma_3^2}_{:=I_2} + \underbrace{3\gamma_1^2\gamma_2\gamma_3}_{:=I_3} + \underbrace{3\gamma_1\gamma_3^3 - \gamma_3^4 - 3\gamma_1^2\gamma_3^2 - 6\gamma_1\gamma_2\gamma_3^2}_{:=I_4} \end{aligned}$$

By Young's inequality, we obtain

$$I_1 \le \tfrac{1}{4}\gamma_3^4 + c\gamma_2^4,\ I_2 \le \tfrac{1}{4}\gamma_1^2\gamma_2^2,\ I_3 \le \gamma_3^2 + c\gamma_1^4\gamma_2^2 \text{ and } I_4 \le -\tfrac{1}{4}\gamma_3^4 + c\gamma_1^2\gamma_2^2.$$

Combining all together, yields

$$\langle \mathcal{F}(\gamma_1 \sin + \gamma_2 \sin - \gamma_3 \sin) - \mathcal{F}(\gamma_1 \sin), \gamma_3 \sin\rangle \leq C[\gamma_2^4 + \gamma_1^2\gamma_2^2 + \gamma_1^4\gamma_2^2 + \gamma_3^2].$$

Now our main theorem states that

$$u(t) = \varepsilon\gamma(\varepsilon^2 t) \sin + \mathcal{O}(\varepsilon^2)$$

where

$$\gamma' = \nu\gamma - \tfrac{1}{12}\gamma^3 + \alpha_1\tilde{\beta}'$$

with a rescaled standard Brownian motion $\tilde{\beta}$.

5.2 Surface growth model

The second example that falls into the scope of our work is the growth of rough amorphous surfaces. The equation is of the type

$$\partial_t h = -\triangle^2 h - \mu\triangle h - \triangle|\nabla h|^2 + \sigma\partial_t W(t). \tag{59}$$

Here $\triangle$ is Laplacian with respect to periodic boundary conditions on $[0, 2\pi]$. Suppose initial condition $h(0) = 0$ corresponding to an initially flat surface.

For this model we consider $\mu = 1 + \varepsilon^2\nu$ and $\sigma = \varepsilon^2$. Hence

$$A = -\triangle^2 - \triangle, \quad L = -\nu\triangle \quad \text{and} \quad B(u,v) = -\triangle(\partial_x u \cdot \partial_x v).$$

We take

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if } k > 0, \\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if } k < 0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k = 0 \end{cases}$$

and

$$\mathcal{H} = \{u \in L^2([0,2\pi]) : \int_0^{2\pi} u dx = 0\} \quad \text{and} \quad \mathcal{N} = \text{span}\{\sin,\ \cos\}.$$

The eigenvalues of $-A = \triangle^2 + \triangle$ are $\lambda_k = k^4 - k^2$ with $m = 4$ and $\lim_{k\to\infty}\lambda_k = \infty$. So, the Assumption 1 is true.

If we define $u(t) := h(t) - h_0(t)e_0$, then we obtain

$$\partial_t u = -\triangle^2 u - \mu\triangle u - \triangle|\nabla u|^2 + \sigma\sum_{k\neq 0}\alpha_k\partial_t\beta_k(t)e_k, \tag{60}$$

and

$$h_0 = \sigma\alpha_0\beta_0(t). \tag{61}$$

If $u = \gamma_1 \sin + \gamma_{-1}\cos \in \mathcal{N}$, then

$$B(u,u) = 2\left[\gamma_1^2 - \gamma_{-1}^2\right]\cos(2x) - 4\gamma_1\gamma_{-1}\sin(2x),$$

and

$$P_c B(u,u) = 0,$$

and for $\alpha = \frac{5}{4}$ and $\beta = \frac{13}{4} < m$, we obtain

$$\begin{aligned}\|B(u,v)\|_{\mathcal{H}^{-2}} &= \|\triangle(\partial_x u \cdot \partial_x v)\|_{\mathcal{H}^{-2}} \leq c\|\partial_x u \cdot \partial_x v\|_{L^2} \\ &\leq c\|u\|_{\mathcal{H}^{\frac{5}{4}}}\|v\|_{\mathcal{H}^{\frac{5}{4}}}.\end{aligned}$$

Hence all conditions of Assumption 6 are satisfied. Moreover, the Assumption 8 is true, where

$$\mathcal{F}(\gamma_1 \sin + \gamma_{-1} \cos) = \tfrac{1}{6}[(\gamma_1^3 + \gamma_1\gamma_{-1}^2) \sin + (\gamma_{-1}^3 + \gamma_{-1}\gamma_1^2) \cos],$$

and

$$\langle \mathcal{F}(u), u\rangle \geq \tfrac{1}{6\pi}\|u\|^4.$$

The amplitude equation for (60) is a system of stochastic ordinary differential equations:

$$d\gamma_i = [\nu\gamma_i - \tfrac{1}{3}\gamma_i(\gamma_1^2 + \gamma_{-1}^2)]dt + \alpha_i d\beta_i \quad \text{for } i = \pm 1.$$

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